Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Geometry and Physics 56 (2006) 670–690

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Hyper-parahermitian manifolds with torsion

Stefan Ivanov*, Vasil Tsanov, Simeon Zamkovoy

University of Sofia "St. Kl. Ohridski", Faculty of Mathematics and Informatics, Blvd. James Bourchier 5, 1164 Sofia, Bulgaria

Received 7 March 2005; received in revised form 14 April 2005; accepted 22 April 2005
Available online 13 June 2005

Abstract

Necessary and sufficient conditions for the existence of a hyper-parahermitian connection with totally skew-symmetric torsion (HPKT-structure) are presented. It is shown that any HPKT-structure is locally generated by a real (potential) function. An invariant first order differential operator is defined on any almost hyper-paracomplex manifold showing that it is two-step nilpotent exactly when the almost hyper-paracomplex structure is integrable. A local HPKT-potential is expressed in terms of this operator. Examples of (locally) invariant HPKT-structures with closed as well as non-closed torsion 3-form on a class of (locally) homogeneous hyper-paracomplex manifolds (some of them compact) are constructed.

MSC: 53C15; 5350; 53C25; 53C26; 53B30

Keywords: Hyper-parahermitian structure; Skew-symmetric torsion; Potential; Paracomplex and paraquaternionic operator

1. Introduction

We study the geometry of structures on a differentiable manifold related to the algebra of paraquaternions together with a naturally associated metric which is necessarily of neutral signature. This structure leads to the notion of (almost) hyper-paracomplex and

* Corresponding author.
E-mail addresses: ivanovsp@fmi.uni-sofia.bg (S. Ivanov), tsanov@fmi.uni-sofia.bg (V. Tsanov), zamkovoy@fmi.uni-sofia.bg (S. Zamkovoy).

0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2005.04.012

hyper-parahermitian manifolds in dimensions divisible by four. These structures are also attractive in theoretical physic since some of them play a role in string theory [27,18,6,19] and integrable systems [9].

Hyper-parahermitian geometry may be interpreted as the indefinite analog of hyper-hermitian geometry, but there are important differences. We provide hyper-parahermitian versions of many local and some global results for hyper-hermitian manifolds, specially we adopt the hyper-complex constructions of [16,14,4] (but see also [30,24,25,33]).

We treat integrable almost hyper-parahermitian structures, which admit compatible linear connections with totally-skew symmetric torsion, briefly HPKT-structure. It is known that in dimension 4, the conformal structure of neutral signature determined by a hyper-paracomplex structure is necessarily anti-self-dual [1,18,21]. We show that the corresponding conformal hyper-parahermitian structure is an HPKT-structure. In higher dimensions, we find necessary and sufficient conditions for the existence of a HPKT-structure in terms of the exterior derivative of the three Kähler forms. We give a holomorphic characterization and show uniqueness of the HPKT-connection.

To illustrate the subtleties of HPKT we use some homogeneous examples and their compact factors found in [20]. In particular, we show the existence of an invariant HPKT-structure with closed torsion 3-form on the simple Lie groups $SU(m, m-1), m > 1$, associated to the biinvariant Killing-Cartan form of neutral signature on $SU(m, m-1)$. In contrast, the HPKT-structures for the hyper-paracomplex structures on the simple Lie groups $SL(2m-1, \mathbb{R}), m > 1$ obtained in [20] have no compatible biinvariant metric. They may be associated to (a class of) invariant metrics of neutral signature, which however have non-closed torsion forms.

We show that any HPKT-structure is locally generated by a real (potential) function following the ideas developed in [4]. To this end, using Salamon's idea from the quaternionic case (see [30]), we define an invariant first order differential operator D, the hyper-paracomplex operator, on an almost hyper-paracomplex manifold and we show that it is 2-step nilpotent exactly when the almost hyper-paracomplex structure is integrable. Then, we obtain the local existence of HPKT-potential by proving the local D-exactness of certain D-closed 2-forms.

2. Hyper-paraKähler connection with torsion

Both quaternions H and paraquaternions $\tilde{H}$ are real Clifford algebras, $H = C(2, 0)$, $\tilde{H} = C(1, 1) \cong C(0, 2)$. In other words, the algebra $\tilde{H}$ of paraquaternions is generated by the unity 1 and the generators J_1, J_2, J_3 satisfying the *paraquaternionic identities*,

$$J_1^2 = J_2^2 = -J_3^2 = 1, \qquad J_1 J_2 = -J_2 J_1 = J_3. \tag{2.1}$$

We recall the notion of almost hyper-paracomplex manifold introduced by Libermann [23]. An *almost quaternionic structure of the second kind* on a smooth manifold consists of two almost product structures J_1, J_2 and an almost complex structure J_3 which mutually anti-commute, i.e. these structures satisfy the paraquaternionic identities (2.1). Such a structure is also called *complex product structure* [3,2].

An *almost hyper-paracomplex structure* on a $4n$-dimensional manifold M is a triple $\tilde{H} = (J_a), a = 1, 2, 3$, where $J_\alpha, \alpha = 1, 2$ are almost paracomplex structures $J_\alpha : TM \to TM$, and $J_3 : TM \to TM$ is an almost complex structure, satisfying the paraquaternionic identities (2.1). We note that on an almost hyper-paracomplex manifold there is actually a 2-sheeted hyperboloid worth of almost complex structures: $S^2_1(-1) = \{c_1 J_1 + c_2 J_2 + c_3 J_3 : c_1^2 + c_2^2 - c_3^2 = -1\}$ and a 1-sheeted hyperboloid worth of almost paracomplex structures: $S^2_1(1) = \{b_1 J_1 + b_2 J_2 + b_3 J_3 : b_1^2 + b_2^2 - b_3^2 = 1\}$.

When each $J_a, a = 1, 2, 3$ is an integrable structure, $\tilde{H}$ is said to be a *hyper-paracomplex structure* on M. Such a structure is also called sometimes *pseudo-hyper-complex* [9]. Any hyper-paracomplex structure admits a unique torsion-free connection ∇^{CP} preserving J_1, J_2, J_3 [3,2] called *the complex product connection*.

The Nijenhuis tensor N_a of J_a is defined by:

$$N_\alpha(X, Y) = [J_\alpha X, J_\alpha Y] + J_\alpha^2[X, Y] - J_\alpha[J_\alpha X, Y] - J_\alpha[X, J_\alpha Y]. \tag{2.2}$$

It is well known that the structure J_a is integrable if and only if the corresponding Nijenhuis tensor N_a vanishes, $N_a = 0$.

In fact an almost hyper-paracomplex structure is hyper-paracomplex if and only if any two of the three structures $J_a, a = 1, 2, 3$ are integrable due to the existence of a linear identity between the three Nijenhuis tensors [21,7]. In this case, all almost complex structures of the two-sheeted hyperboloid $S^2_1(-1)$ as well as all paracomplex structures of the one-sheeted hyperboloid $S^2_1(1)$ are integrable.

A *hyper-parahermitian metric* is a pseudo Riemannian metric which is compatible with the (almost) hyper-paracomplex structure $\tilde{H} = (J_a), a = 1, 2, 3$ in the sense that the metric g is skew-symmetric with respect to each $J_a, a = 1, 2, 3$, i.e.,

$$g(J_1., J_1.) = g(J_2., J_2.) = -g(J_3., J_3.) = -g(., .). \tag{2.3}$$

The metric g is necessarily of neutral signature $(2n,2n)$. Such a structure is called *(almost) hyper-parahermitian structure*.

Let F_a be the Kähler form associated with the structure $(g, J_a), a = 1, 2, 3$:

$$F_a = g(., J_a.).$$

The corresponding Lee form is defined by $\theta_a = -\delta F_a \circ J_a^3$. In particular,

$$\theta_a(X) = \sum_{i=1}^{2n} dF_a(e_i, J_a e_i, J_a^2 X),$$

for an orthonormal J_a-adapted basis $\{e_1, \ldots, e_{2n}, J_a e_1, \ldots, J_a e_{2n}\}$.

If on a hyper-parahermitian manifold there exists an admissible basis $(\tilde{H})$ such that each $J_a, a = 1, 2, 3$ is parallel with respect to the Levi-Civita connection or equivalently the three Kähler forms are cosed, $dF_a = 0$ then the manifold is called *hyper-paraKähler*. Such manifolds are also called *hypersymplectic* [15], *neutral hyper-Kähler* [22,10]. The equivalent characterization is that the holonomy group of g is contained in $Sp(n, \mathbb{R})$ if $n \geq 2$ [34].

For $n = 1$ an (local) almost hyper-paracomplex structure is the same as oriented neutral conformal structure [9,12,34,7]. The existence of a (local) hyper-paracomplex structure is a strong condition since the integrability of the (local) almost hyper-paracomplex structure implies that the corresponding neutral conformal structure is anti-self-dual [1,18,21]. The necessary and sufficient condition for the integrability of an (local) almost hyper-paracomplex structure in dimension four is the coincidence of the three Lee forms, $\theta_1 = \theta_2 = \theta_3$ [21].

We use the following notations: for any r-form ω we define $J_a\omega(X_1, \dots, X_r) := (-1)^r\omega(J_aX_1, \dots, J_aX_r), a = 1, 2, 3$ and the operators $d_\alpha\omega := -J_\alpha dJ_\alpha\omega, \alpha = 1, 2, d_3\omega := (-1)^r J_3dJ_3\omega$. In particular, $d_aF_a = -dF_a(J_a., J_a., J_a),\ a = 1, 2, 3$.

We consider the (para) complex operators:

$$\partial_\alpha = \tfrac{1}{2}(d + \epsilon d_\alpha), \quad \overline{\partial}_\alpha = \tfrac{1}{2}(d - \epsilon d_\alpha), \quad \epsilon^2 = 1, \quad \alpha = 1, 2$$
$$\partial_3 = \tfrac{1}{2}(d + id_3), \quad \overline{\partial}_3 = \tfrac{1}{2}(d - id_3), \quad i^2 = -1.$$

In particular, a complex function $f = u + iv$ is holomorphic with respect to the complex structure J_3 iff $\overline{\partial}_3 f = 0$ while a paracomplex function $h = u + \epsilon v$ is paraholomorphic with respect to the paracomplex structure $J_\alpha, \alpha = 1, 2$ iff $\overline{\partial}_\alpha h = 0$.

Definition 2.1. A hyper-parahermitian metric g is *hyperparaKähler with torsion* (briefly HPKT) if there exists a linear connection ∇ preserving the hyper-paracomplex structure whose torsion tensor T^∇ is totally skew-symmetric, i.e.,

$$\nabla g = \nabla J_1 = \nabla J_2 = \nabla J_3 = 0,$$
$$T^\nabla(X, Y, Z) := g(T^\nabla(X, Y), Z) = -T^\nabla(X, Z, Y). \tag{2.4}$$

If the torsion 3-form T^∇ is closed, $dT^\nabla = 0$, then the HPKT-metric is called *strong HPKT-metric*.

A connection satisfying condition (2.4) will be called briefly *HPKT-connection*.

Theorem 2.2. *Let (M, g, J_1, J_2, J_3) be a hyper-parahermitian manifold. The following conditions are equivalent*:

(1) (M, g, J_1, J_2, J_3) *admits a HPKT-connection*;
(2) *The following equalities hold*:

$$d_1F_1 = d_2F_2 = d_3F_3. \tag{2.5}$$

In this case, the HPKT-connection is uniquely determined by the torsion:

$$T^\nabla = d_1F_1 = d_2F_2 = d_3F_3. \tag{2.6}$$

In particular, the three Lee forms coincide, $\theta_1 = \theta_2 = \theta_3 = tr_gT^\nabla$.

Proof. The required connection is the unique Bismut connection determined by Gauduchon [13] (see also [11]) in the hermitian case and by Ivanov–Zamkovoy [21] in the parahermitian case due to the compatibility condition (2.5). Take the trace in (2.6) to get the last identity. □

It is known that in dimension four any hyper-parahermitian metric is anti-self-dual [1,18,21]. The proof of Theorem 6.2 in [21] leads to:

Proposition 2.3. *Any hyper-parahermitian metric on a hyper-paracomplex 4-manifold is HPKT. In particular, the Ricci 2-forms of the HPKT-connection all vanish.*

3. Homogeneous examples

A non-trivial class of examples for the differential geometric entities defined in the previous section is provided by certain left-invariant HPKT-structures on (semi) simple Lie groups which were found in [20]. For convenience we reproduce here the explicit description of J_2 and J_3. We define the (para-) complex structures on Lie algebras and then interpret them as homogeneous almost (para-) complex structures on the corresponding simply connected Lie groups. For brevity we shall sometimes abuse notation and proper definitions, by indicating only the Lie algebras.

3.1. HPKT-structure on $SU(m, m-1)$

The most important example from [20]is the group $SU(m, m-1)$, where the biinvariant Killing form is the neutral HPKT-metric.

On the simple Lie algebra $su(m, m-1)$ (of dimension $4m(m-1)$) we define a scalar product:

$$B(X, Y)\doteq\frac{1}{2}tr(XY), \quad X, Y \in su(m, m-1) \tag{3.7}$$

Obviously B is proportional to the Killing form and defines a biinvariant, neutral pseudoriemannian metric on $SU(m, m-1)$. Next we produce a convenient B - orthonormal base of the Lie algebra $su(m, m-1)$. As usual, we denote by $E_j^k \in gl(n)$ the matrix with entry 1 at the intersection of the j-th row and the k-th column and 0 elsewhere. We fix the range of indices:

$$j = 1, \dots, m-1, \qquad j < k < 2m-j. \tag{3.8}$$

Let $\mathfrak{z}$ be the subspace (abelian subalgebra) of $su(m, m-1)$ generated by the elements

$$i(E_j^j + E_{2m-j}^{2m-j} - 2E_m^m), \quad j = 1, \dots, m-1.$$

Let $Z^1, \dots Z^{m-1}$ be any orthonormal base of $\mathfrak{z}$, with respect to the scalar product[1] B. We define:

$$X^j\doteq i(E_j^j - E_{2m-j}^{2m-j}); \quad Y^j\doteq E_j^{2m-j} + E_{2m-j}^j : \quad W^j\doteq i(E_j^{2m-j} - E_{2m-j}^j). \tag{3.9}$$

[1] B is obviously negative definite on $\mathfrak{z}$.

$$U = U_j^k \doteq \begin{cases} E_j^k - E_k^j & \text{if } j < k \le m; \\ E_k^{2m-j} - E_{2m-j}^k & \text{if } m < k < 2m-j. \end{cases}$$
$$V = V_j^k \doteq \begin{cases} i(E_j^k + E_k^j) & \text{if } j < k \le m; \\ i(E_k^{2m-j} + E_{2m-j}^k) & \text{if } m < k < 2m-j. \end{cases}$$
$$S = S_j^k \doteq \begin{cases} E_k^{2m-j} + E_{2m-j}^k & \text{if } j < k \le m; \\ E_j^k - E_k^j & \text{if } m < k < 2m-j. \end{cases} \tag{3.10}$$
$$T = T_j^k \doteq \begin{cases} i(E_k^{2m-j} - E_{2m-j}^k) & \text{if } j < k \le m; \\ i(E_k^j - E_j^k) & \text{if } m < k < 2m-j. \end{cases}$$

The invariant vector fields (generated by) $X^j, Y^j, W^j, Z^j, U_j^k, V_j^k, S_j^k, T_j^k$ give a base of the tangent bundle of $SU(m, m-1)$. We define an almost hyper-paracomplex structure by:

$$\begin{aligned} &J_3(Z^j) \doteq X^j; \quad J_3(Y^j) \doteq W^j; \quad J_2(Z^j) \doteq W^j, \quad J_2(X^j) \doteq Y^j; \\ &J_3(U_j^k) \doteq V_j^k; \quad J_3(S_j^k) \doteq T_j^k; \quad J_2(U_j^k) \doteq T_j^k; \quad J_2(V_j^k) \doteq S_j^k \end{aligned} \tag{3.11}$$

It is shown in [20] that the structure (3.11) is an integrable hyper-paracomplex structure on $SU(m, m-1), m > 1$ which is compatible with the biinvariant (Killing-Cartan) form of neutral signature B.

Now we observe, that the above construction gives also a strong HPKT-structure on $SU(m, m-1)$. The HPKT-connection is the left-invariant connection ∇, defined by postulating all left-invariant vector fields to be parallel.

The torsion of the above connection is the Lie bracket and the torsion tensor $T^\nabla(X, Y, Z) = -B([X, Y], Z)$ is a closed 3-form (due to the Jacobi identity). So, we have a strong HPKT-structure on $SU(m, m-1)$ which is flat. The compatible neutral Killing-Cartan metric is Einstein.

Simple Lie groups admit cocompact lattices [8], say Γ. Hence, we obtain a HPKT-interpretation of the result proved in [20].

Theorem 3.1. [20] *The compact manifolds $SU(m, m-1)/\Gamma$ admit invariant, flat, strong HPKT-structures. The neutral HPKT-metric is a non-flat Einstein metric induced by the Killing-Cartan form.*

Remark 3.2. The above procedure can be applied to the group $(SL(2m-1, \mathbb{C}))^{\mathbb{R}}$ (see [20]). Thus, we obtain invariant strong and flat HPKT-structures on the compact manifolds $(SL(2m-1, \mathbb{C}))^{\mathbb{R}}/\Gamma$.

3.1.1. A non-strong HPKT-structure on $SU(2, 1)$

We equipped the eight-dimensional simple Lie group $SU(2, 1)$ with a strong and flat left-invariant HPKT-structure induced by the left-invariant hyper-paracomplex structure (3.11) and the Killing-Cartan form. We show below that a similar[2] hyper-paracomplex structure supports left-invariant HPKT-structure which is not strong and not flat . To be precise, we consider the following base on $su(2,1)$:

$$\begin{aligned} &Z\doteq i(E_1^1+E_3^3-2E_2^2); && X\doteq i(E_1^1-E_3^3);\\ &W\doteq i(E_1^3-E_3^1); && Y\doteq E_1^3+E_3^1;\\ &U\doteq E_1^2-E_2^1; && V\doteq i(E_1^2+E_2^1);\\ &S\doteq E_2^3+E_3^2; && T\doteq i(E_2^3-E_3^2). \end{aligned} \tag{3.12}$$

A hyper-paracomplex structure on $SU(2,1)$ is given by:

$$\begin{aligned} &J_3(Z)\doteq X; \quad J_3(Y)\doteq W; \quad J_2(Z)\doteq W, \quad J_2(X)\doteq Y;\\ &J_3(U)\doteq V; \quad J_3(S)\doteq T; \quad J_2(U)\doteq T; \quad J_2(V)\doteq S, \end{aligned} \tag{3.13}$$

We claim that the neutral metric g determined by the following orthonormal basis:

$$\begin{aligned} &g(Z,Z)=g(X,X)=g(U,U)=g(V,V)=1,\\ &g(Y,Y)=g(W,W)=g(S,S)=g(T,T)=-1 \end{aligned} \tag{3.14}$$

is a non strong left-invariant HPKT-metric on $SU(2,1)$ with respect to the left-invariant hyper-paracomplex structure (3.13). We denote the 1-form dual to a vector field via the neutral metric (3.14) by the same letter. We calculate

$$\begin{aligned} T^{\nabla} &= d_1F_1=d_2F_2=d_3F_3\\ &=2X\wedge Y\wedge W-X\wedge U\wedge V+X\wedge S\wedge T+Y\wedge U\wedge S-Y\wedge V\wedge T\\ &\quad +W\wedge U\wedge T+W\wedge V\wedge S-Z\wedge U\wedge V-Z\wedge S\wedge T;\\ dT^{\nabla} &= -4U\wedge V\wedge S\wedge T\neq 0. \end{aligned}$$

Our claim is proved.

3.2. HPKT-structure on $SL(2m-1,\mathbb{R})$

Consider the simple Lie group $SL(2m-1,\mathbb{R})$ with the almost hyper-paracomplex structure (3.11) applied to the following base[3] of $sl(2m-1,\mathbb{R})$:

$$\begin{aligned} &Z^j\doteq E_j^j+E_{2m-j}^{2m-j}-2E_m^m; && W^j\doteq E_j^j-E_{2m-j}^{2m-j};\\ &X^j\doteq E_j^{2m-j}-E_{2m-j}^j; && Y^j\doteq E_j^{2m-j}+E_{2m-j}^j;\\ &U_j^k\doteq E_j^k-E_k^j; && V_j^k\doteq E_k^{2m-j}-E_{2m-j}^k;\\ &S_j^k\doteq E_k^{2m-j}+E_{2m-j}^k; && T_j^k\doteq E_j^k+E_k^j. \end{aligned} \tag{3.15}$$

[2] We choose the simplest one in a notational sense. Obviously, any neutral metric on the Lie algebra $\mathfrak{g}$, which is compatible with a paraquaternionic structure, gives a left invariant metric on the corresponding Lie group G. However, there is only one biinvariant metric on a simple group.

[3] The range of indices is as in formula (3.8).

The structure (3.11) is an integrable left-invariant hyper-paracomplex structure on $SL(2m-1,\mathbb{R})$, $m>1$ [20].

A left-invariant neutral metric g, determined by the following orthonormal basis of $sl(2m-1,\mathbb{R})$:

$$\begin{aligned} &g(Z^j,Z^j)=g(X^j,X^j)=g(U_j^k,U_j^k)=g(V_j^k,V_j^k)=1,\\ &g(Y^j,Y^j)=g(W^j,W^j)=g(S_j^k,S_j^k)=g(T_j^k,T_j^k)=-1 \end{aligned} \tag{3.16}$$

is an HPKT-metric on $sl(2m-1,\mathbb{R})$, $m>1$ with respect to the hyper-paracomplex structure (3.11), which has not closed torsion 3-form. Indeed, denoting the 1-form dual to a vector field via the neutral metric (3.16) by the same letter, we calculate

$$\begin{aligned} T^\nabla &= d_1F_1=d_2F_2=d_3F_3\\ &=2X^j\wedge Y^j\wedge W^j - X^j\wedge U_j^k\wedge V_j^k + X^j\wedge S_j^k\wedge T_j^k + Y^j\wedge U_j^k\wedge S_j^k\\ &\quad - Y^j\wedge V_j^k\wedge T_j^k + W^j\wedge U_j^k\wedge T_j^k + W^j\wedge V_j^k\wedge S_j^k + Z^j\wedge U_j^k\wedge T_j^k\\ &\quad - Z^j\wedge V_j^k\wedge S_j^k;\\ dT^\nabla &= -8U_j^k\wedge V_j^k\wedge S_j^k\wedge T_j^k\neq 0. \end{aligned}$$

The groups $SL(2m-1,\mathbb{R})$ admit cocompact lattices [8], say Γ. Thus, we arrive at a HPKT-extension of the results in [20].

Theorem 3.3. *The compact manifolds $SL(2m-1,\mathbb{R})/\Gamma$ admit an HPKT-structure which are not strong.*

3.3. *HPKT-structure on* $2\mathbb{R}\oplus sl(2,\mathbb{C})$

We consider the following base on $2\mathbb{R}\oplus sl(2,\mathbb{C})$:

$$\begin{aligned} &Z\doteq\begin{bmatrix}1&0\\0&1\end{bmatrix};\qquad X\doteq\begin{bmatrix}0&1\\-1&0\end{bmatrix};\qquad Y\doteq\begin{bmatrix}0&1\\1&0\end{bmatrix};\qquad W\doteq\begin{bmatrix}1&0\\0&-1\end{bmatrix}\\ &U\doteq iZ;\qquad V\doteq iX;\qquad S\doteq iY;\qquad T\doteq iW. \end{aligned} \tag{3.17}$$

We define an almost hyper-paracomplex structure on the Lie algebra $2\mathbb{R}\oplus sl(2,\mathbb{C})\cong 2\mathbb{R}\oplus so(3,1)$ by (3.13) using the base (3.17) . It is easy to check that this structure is integrable.

We claim that the neutral metric g determined by the orthonormal basis (3.14) is a strong left-invariant HPKT-metric on the simply connected Lie group G associated to the Lie algebras $2\mathbb{R}\oplus sl(2,\mathbb{C})\cong 2\mathbb{R}\oplus so(3,1)$ with respect to the left-invariant hyper-paracomplex structure (3.13). To prove the claim, we denote the 1-form dual to a vector field via the neutral metric (3.14) by the same letter. We obtain

$$T^\nabla=d_1F_1=d_2F_2=d_3F_3=S\wedge dS-Y\wedge dY;\qquad dT^\nabla=0$$

which proves our claim.

Let $\Delta \subset SL(2, \mathbb{C})$ be a cocompact discrete subgroup and let $\Gamma = \mathbb{Z} \times \mathbb{Z} \times \Delta \subset G$. We obtain:

Theorem 3.4. *The compact manifold G/Γ admits a strong HPKT-structure.*

4. Characterization of HPKT-structures

In this section, we characterize HPKT-structures in terms of the existence of holomorphic objects. We use ideas from the definite (HKT) case described in [16,14] and find other compact examples.

4.1. Holomorphic characterization

We recall that the space of paracomplex (1, 0)-vectors (resp. (0, 1)-vectors) with respect to the paracomplex structure $J_\alpha, \alpha = 1, 2$ is spanned by paracomplex vectors of type $X + \epsilon J_\alpha X$ (resp. $X - \epsilon J_\alpha X$) and the space of complex (1, 0)-vectors (resp. (0, 1)-vectors) of the complex structure J_3 is spanned as usual by complex vectors of type $X - iJ_3X$ (resp. $X + iJ_3X$).

It is easy to check that

: the 2-form $F_2 - \epsilon F_3$ is of type (2, 0) while the 2-form $F_2 + \epsilon F_3$ is of type (0, 2) with respect to the paracomplex structure J_1;
: the 2-form $F_3 + \epsilon F_1$ is of type (2, 0) while the 2-form $F_3 - \epsilon F_1$ is of type (0, 2) with respect to the paracomplex structure J_2;
: the 2-form $F_1 - iF_2$ is of type (2, 0) while the 2-form $F_1 + iF_2$ is of type (0, 2) with respect to the complex structure J_3;

Proposition 4.1. *Let $(M, g, J_a, a = 1, 2, 3)$ be a hyper-parahermitian manifold. The following conditions are equivalent*:

(a) $(M, g, J_a, a = 1, 2, 3)$ *is a PHKT manifolld*;
(b) $\partial_1(F_2 - \epsilon F_3) = 0$ *or equivalently* $\underline{\partial}_1(F_2 + \epsilon F_3) = 0$;
(c) $\partial_2(F_3 + \epsilon F_1) = 0$ *or equivalently* $\underline{\partial}_2(F_3 - \epsilon F_1) = 0$;
(d) $\partial_3(F_1 - iF_2) = 0$ *or equivalently* $\bar{\partial}_3(F_1 + iF_2) = 0$.

Proof. We have:

$$\partial_1(F_2 - \epsilon F_3) = \bar{\partial}_1(F_2 + \epsilon F_3) = \frac{1}{2}(dF_2 - d_1F_3) - \frac{\epsilon}{2}(dF_3 - d_1F_2).$$

Therefore $\partial_1(F_2 - \epsilon F_3) = 0$, when the real and imaginary parts both vanishes. We calculate

$$\begin{aligned} d_1F_3 &= -J_1dJ_1F_3 = -J_1d(F_3 \circ J_1) = -J_1dF_3 = (dF_3 \circ J_1) = (dF_3 \circ J_3J_2) \\ &= -J_3(dF_3 \circ J_2) = J_2J_3dF_3 = J_2J_3dJ_3F_3 = J_2d_3F_3. \end{aligned}$$

On the other hand

$$dF_2 = -d(F_2 \circ J_2) = -J_2^2 dJ_2 F_2 = J_2 d_2 F_2.$$

Consequently, the condition $d_1 F_3 = dF_2$ is equivalent to the condition $d_2 F_2 = d_3 F_3$. Therefore, the Bismut connection of the parahermitian structure (g, J_2) coincides with Bismut connection of the hermitian structure (g, J_3). Since $J_1 = J_3 J_2$ then J_1 is parallel with respect to the common connection ∇. Therefore, ∇ is the Bismut connection for (g, J_1) which proves the equivalence of (a) and (b). In a similar way, one completes the proof. □

Proposition 4.1 implies that the HPKT-condition is not preserved by a generic conformal transformation of the metric provided the dimension is at least eight.

In the proof of Proposition 4.1, we also derive

Corollary 4.2. *Suppose F_1, F_2 and F_3 are the Kähler forms of a hyper-parahermitian structure. Then the hyper-parahermitian structure is HPKT-structure if and only if*

$$d_a F_b = \delta_{ab} T^{\nabla} - \epsilon_{abc} F_c,$$

where δ_{ab} is the Kroneker symbol and ϵ_{abc} is the totally skew-symmetric Levi-Civita symbol.

Theorem 4.3. *Let $(M, J_a, a = 1, 2, 3)$ be a hyper-paracomplex manifold. Then any one of the following three conditions implies the forth*:

(1) *$F_2 + \epsilon F_3$ is a $(0, 2)$-form with respect to J_1 such that $\bar{\partial}_1(F_2 + \epsilon F_3) = 0$ and $F_2(X, J_2 Y) = g(X, Y)$ is a symmetric non-degenerate bilinear form of neutral signature*;
(2) *$F_3 - \epsilon F_1$ is a $(0, 2)$-form with respect to J_2 such that $\bar{\partial}_2(F_3 - \epsilon F_1) = 0$ and $F_1(X, J_1 Y) = g(X, Y)$ is a symmetric non-degenerate bilinear form of neutral signature*;
(3) *$F_1 + iF_2$ is a $(0, 2)$-form with respect to J_3 such that $\bar{\partial}_3(F_1 + iF_2) = 0$ and $F_3(X, J_3 Y) = g(X, Y)$ is a symmetric non-degenerate bilinear form of neutral signature*;
(4) *The structure $(g, J_a, a = 1, 2, 3)$ is a PHKT structure.*

Proof. In view of the Proposition 4.1, it suffices to prove that the metric g is hyper-parahermitian.

Using the fact that $F_2 + \epsilon F_3$ is of type $(0, 2)$ with respect to J_1. Since $X + \epsilon J_1 X$ is of type $(1, 0)$ with respect to J_1, $(F_2 + \epsilon F_3)(X + \epsilon J_1 X, Y) = 0$, for any vectors X, Y. It is equivalent to the identity $F_3(X, Y) = -F_2(J_1 X, Y)$. Then,

$$F(X, J_3 Y) = -F_2(J_1 X, J_3 Y) = -F_2(J_1 X, J_1 J_2 Y) = -F_2(X, J_2 Y) = -g(X, Y).$$

So $F_3(J_3 X, J_3 Y) = F_3(X, Y)$ and g is hermitian with respect to J_3. Since the metric is parahermitian with respect to J_2 and $J_1 = J_3 J_2$, g is parahermitian with respect to J_1.

Similarly, one get the other assertions. □

4.2. HPKT-structures on compact Nilmanifolds

In this section, we construct further examples of homogeneous HPKT-structures, now on some (compact) nilmanifolds.

Let $\{X_1, \ldots, X_{2n}, Y_1, \ldots, Y_{2n}, Z\}$ be a basis for $\mathbb{R}^{4n+1}$. Define commutators by: $[X_j, Y_j] = Z$, all others being zero. These commutators give $\mathbb{R}^{4n+1}$ the structure of the *Heisenberg Lie agebra* $\mathfrak{h}_{2n}$. Let $\mathbb{R}^3$ be the three-dimensional abelian algebra. The direct sum $\mathfrak{n} = \mathfrak{h}_{2n} \oplus \mathbb{R}^3$ is a 2-step nilpotent algebra whose center is four-dimensional. Fix a basis $\{E_1, E_2, E_3\}$ for $\mathbb{R}^3$ and consider the following endomorphisms of $\mathfrak{n}$:

$$J_2 : X_{2j-1} \to Y_{2j}, \quad X_{2j} \to Y_{2j-1} \quad Z \to E_2, \quad E_1 \to -E_3;$$

$$J_3 : X_{2j-1} \to X_{2j}, \quad Y_{2j-1} \to Y_{2j} \quad Z \to E_1, \quad E_2 \to E_3;$$

$$J_2^2 = -J_3^2 = identity, \quad J_1 = J_3 J_2. \tag{4.18}$$

Clearly, $J_2 J_3 = -J_3 J_2$. The almost complex structure J_3 satisfies the identity $[J_3., J_3.] = [., .]$ which implies that it is an Abelian almost complex structure on $\mathfrak{n}$ in the sense of [5] and in particular integrable. The next notion seems to be useful.

Definition 4.4. The almost paracomplex structure J_2 is said to be *Abelian* if the following identity $[J_2., J_2.] = -[., .]$ holds.

Applying (2.2) it is easy to check that any Abelian almost paracomplex structure has vanishing Nijenuis tensor and therefore is integrable. It is easy to verify that the almost paracomplex structure J_2 is Abelian on $\mathfrak{n}$. Consequently, the almost paracomplex structure J_1 is also Abelian. Hence, the structure $J_a, a = 1, 2, 3$ is a left invariant hyper-paracomplex structure on the simply connected Lie group N whose Lie algebra is $\mathfrak{n}$. Consider the invariant metric g on N for which the basis $\{X_j, Y_j, Z, E_a\}$ is orthonormal and $g(X_j, X_j) = g(Z, Z) = g(E_1, E_1) = 1, \quad g(Y_j, Y_j) = g(E_2, E_2) = g(E_3, E_3) = -1$. Clearly, the structure $(g, J_a, a = 1, 2, 3)$ is a left invariant hyper-parahermitian structure on N which turns out to be a HPKT since any left invariant (2, 0)-form with respect to the complex structure J_3 is ∂_3-closed due to a result of Salamon [31] and Proposition 4.1. Because N is isomorphic to the product $H_{2n} \times \mathbb{R}^3$ of the Heisenbrg group H_{2n} and the Abelian group $\mathbb{R}^3$ we have:

Corollary 4.5. *Let Γ be a cocompact lattice in the Heisenberg group H_{2n} and $\mathbb{Z}^3$ a lattice in $\mathbb{R}^3$. The compact Nilmanifold $N/(\Gamma \times \mathbb{Z}^3)$ admits an HPKT-structure.*

4.3. HPKT-structure on $(H_{2n} \times \widetilde{SL(2, \mathbb{R})})/\Gamma$

Based on the above computations, we can also see that there is a left-invariant HPKT-structure on the product of $4n + 1$-dimensional Heisenberg group H_{2n} and the universal cover $\widetilde{SL(2, \mathbb{R})}$ of the simple Lie group $SL(2, \mathbb{R})$. The Lie algebra $sl(2, \mathbb{R})$ has a basis $\{E_1, E_2, E_3\}$ with non-zero brackets given by:

$$[E_1, E_2] = E_3, \quad [E_2, E_3] = -E_1, \quad [E_3, E_1] = E_2.$$

The construction of the HPKT-structure on the product $H_{2n} \times \widetilde{SL(2,\mathbb{R})}$ is the same as those in Section 4.2 taking E_1, E_2, E_3 to be the generators of $sl(2,\mathbb{R})$. The integrability of the (non-abelian) almost hyper-paracomplex structure defined by (4.18) as well as the HPKT-compatibility conditions (2.5) can be checked directly using the commutators of the left-invariant vector fields. Denote the left-invariant 1-forms dual to the left invariant vector fields via the metric by the same letters to get:

$$T^{\nabla} = d_1 F_1 = d_2 F_2 = d_3 F_3 = dZ \wedge Z; \qquad dT^{\nabla} = dZ \wedge dZ \neq 0.$$

The last equalities imply that the HPKT-structure is not strong.

Let Γ_1 be a cocompact lattice in the Heisenberg group H_{2n}. The universal cover $\widetilde{SL(2,\mathbb{R})}$ of the Lie group $SL(2,\mathbb{R})$ admits a discrete subgroup Γ_2 such that the quotient space $(\widetilde{SL(2,\mathbb{R})}/\Gamma_2)$ is a compact 3-manifold [26,29,32]. Such a space has to be Seifert fibre space [32] and all the quotients are classified in [29]. We obtain:

Corollary 4.6. *The compact manifold* $(H_{2n} \times \widetilde{SL(2,\mathbb{R})})/(\Gamma_1 \times \Gamma_2)$ *admits a non-strong HPKT-structure.*

5. Potential theory

It is well known that a Kähler metric is locally generated by a potential ie a real function μ satisfying $F_3 = -dd_3\mu$. Similarly, a paraKäler metric is locally generated by a potential, ie a real function ν satisfying $F_1 = dd_1\nu$ [28].

A function μ is a potential function for a hyper-paraKähler manifold (M, g, J_a) if the Kähler forms are equal to:

$$F_a = J_a^2 dd_a\mu, \quad d_a\mu = -J_a^3 d\mu. \tag{5.19}$$

In this section, we seek a function that generates all Kähler forms of a HPKT-manifold.

The definition of the operators d_a, paraquaternionic identities (2.1), the compatibility conditions (2.3) and (5.19) imply:

$$\begin{aligned} d_1d_2\mu &= -d_1J_2d\mu = J_1dJ_1J_2d\mu = J_1dd_3\mu = -J_1F_3 = -d_2d_1\mu = dd_3\mu;\\ d_2d_3\mu &= d_2J_3d\mu = J_2dJ_1d\mu = -J_2dd_1\mu = -J_2F_1 = -d_3d_2\mu = -dd_1\mu;\\ d_3d_1\mu &= -d_3J_1d\mu = J_3dJ_3J_1d\mu = J_3dd_2\mu = J_3F_2 = -d_1d_3\mu = -dd_2\mu. \end{aligned} \tag{5.20}$$

We generalize this concept to HPKT-manifold.

Definition 5.1. Let (M, g, J_a) be a HPKT-structure with Kähler forms F_1, F_2 and F_3. A possibly locally defined function μ is a potential function for the HPKT-structure if:

$$F_1 = \tfrac{1}{2}(dd_1 - d_2d_3)\mu, \qquad F_2 = \tfrac{1}{2}(dd_2 - d_3d_1)\mu, \qquad F_3 = -\tfrac{1}{2}(dd_3 + d_1d_2)\mu. \tag{5.21}$$

In fact any one of the above identities implies the others due to the next.

Theorem 5.2. *Let (M, g, J_a) be a HPKT-structure with Kähler forms F_1, F_2 and F_3. Let ∇^{CP} be the complex product connection. A possibly locally defined function μ is a potential function for the HPKT-structure if any one of the following identities hold*:

$$F_1 = \frac{1}{2}(dd_1 - d_2d_3)\mu, \tag{5.22}$$

$$F_2 = \frac{1}{2}(dd_2 - d_3d_1)\mu, \tag{5.23}$$

$$F_3 = -\frac{1}{2}(dd_3 + d_1d_2)\mu, \tag{5.24}$$

$$g = \frac{1}{2}(1 - J_1 - J_2 + J_3)(\nabla^{CP})^2\mu. \tag{5.25}$$

The torsion 3-form T^∇ is given by $T^\nabla = -\frac{1}{2}d_1d_2d_3\mu$.

Proof. We calculate, using the fact that the complex product connection is a torsion-free and preserves the hyper-paracomplex structure, that

$$dd_3\mu(X, Y) = -(\nabla^{CP}_X d\mu)J_3Y + (\nabla^{CP}_Y d\mu)J_3X;$$

$$d_1d_2\mu(X, Y) = J_1dd_3\mu(X, Y) = (\nabla^{CP}_{J_1X} d\mu)J_2Y - (\nabla^{CP}_{J_1Y} d\mu)J_2X;$$

$$g(X, Y) = -F_3(X, J_3Y) = \frac{1}{2}(dd_3 + d_1d_2)\mu = \frac{1}{2}(1 - J_1 - J_2 + J_3)(\nabla^{CP})^2\mu$$

Thus, the equivalence of (5.24) and (5.25) is proved.

Similarly, one can get the equivalence of (5.22) and (5.25) as well as the equivalence between (5.23) and (5.25).

The formula for the torsion is a consequence of (5.21) and (2.6). □

Remark 5.3. In the context of a potential, an HPKT-structure is hyper-paraKähler if and only if the potential function μ satisfies any of the following four identities:

$$dd_1\mu = -d_2d_3\mu, \qquad dd_2\mu = -d_3d_1\mu, \qquad dd_3\mu = d_1d_2\mu,$$

$$(1 + J_1 + J_2 + J_3)(\nabla^{CP})^2\mu = 0.$$

Corollary 5.4. *Let (M, g, J_a) be a HPKT-structure with Kähler forms F_1, F_2 and F_3. A possibly locally defined function μ is a potential function for the HPKT-structure if any one of the following identities hold*:

$$F_2 - \epsilon F_3 = -2\partial_1 J_2 \overline{\partial}_1\mu; \tag{5.26}$$

$$F_3 + \epsilon F_1 = -2\partial_2 J_3 \overline{\partial}_2\mu; \tag{5.27}$$

$$F_1 - iF_2 = -2\partial_3 J_1 \overline{\partial}_3\mu. \tag{5.28}$$

Proof. Due to (5.20) and the definition of the operators ∂_a, $\overline{\partial}_a$, we have:

$$F_1 - iF_2 = \frac{1}{2}(dd_1 - d_2d_3 - idd_2 + id_3d_1) = -2\partial_3 J_1 \overline{\partial}_3\mu$$

The other assertions follow in a similar way. □

5.1. HPKT-potential in dimension 4

Now we give a hyperbolic version of the existence of the HKT potentials of Banos and Swann [4]. We apply Theorem 5.2 to the four-dimensional case to prove the existence of a local HPKT-potential for any HPKT-metric.

Corollary 5.5. *Let g be an HPKT-metric on a four-dimensional hyper-paracomplex manifold and let θ be 1-form defined by the complex product connection via $\nabla^{CP} g = \theta \otimes g$.*

A function μ is an HPKT-potential for g if and only if it is solution of the hyperbolic equation

$$\triangle\mu - d\mu(\theta^{\sharp}) + 2 = 0,$$

where $\triangle$ is the hyperbolic Laplacian of the neutral metric g.

In particular, any HPKT-metric on a four-dimensional hyper-paracomplex manifold admits locally a potential.

Proof. Let $A = \nabla^{CP} - \nabla^{g}$, where ∇^{g} denote the Levi-Civita connection of g. The tensor A is symmetric, $A(X, Y) = A(Y, X)$ since both connections are torsion-free. We also have

$$\theta(X)g(Y, Z) = -g(A(X, Y), Z) - g(A(X, Z), Y).$$

Solving for A, we obtain

$$g(A(X, Y), Z) = \tfrac{1}{2}(\theta(Z)g(X, Y) - \theta(X)(g(Y, Z) - \theta(Y)g(Y, Z))).$$

In particular, if X is a (local) unit vector field, then

$$g(A(X, X), Y) = \tfrac{1}{2}\theta(Y) - \theta(X)g(X, Y)$$

and

$$g(A(X, X) - A(J_1X, J_1X) - A(J_2X, J_2X) + A(J_3X, J_3X), Y) = \theta(Y)$$

for all Y.

The metric g is the unique hyper-parahermitian metric satisfying $g(X, X) = 1$. Therefore, μ is a HPKT-potential if and only if

$$\tfrac{1}{2}(1 - J_1 - J_2 + J_3)(\nabla^{CP})^2(X, X) = 1,$$

that is

$$Trace_g(\nabla^{CP} d\mu) = 2.$$

Note that the hyperbolic Laplacian $\triangle\mu$ is by definition $-Trace_g(\nabla^g d\mu)$. Thus, μ is a HPKT-potential for g if and only if

$$-\triangle\mu + d\mu(A(X,X) - A(J_1X, J_1X) - A(J_2X, J_2X) + A(J_3X, J_3X)) = 2.$$

The local existence of HPKT-potentials now follows from the general theory for the (ultra) hyperbolic Laplace operator (see e.g. [17] and the references therein). □

5.2. *HPKT-potential in dimension* $4n \geq 8$

Here we demonstrate the existence of a local potential for any HPKT-metric (the HKT case was done by Banos and Swann [4]).

The crucial step is the construction on any almost hyperparacomplex manifold of an $GL(n, \tilde{H})$-invariant first order differential operator D, which is the hyperbolic version of the hypercomplex (quaternionic) operator of Salamon (see [30]). The operator D is two-step nilpotent if and only if the structure is hyperparacomplex. Then, we obtain the existence of a local HPKT-potential in terms of the operator D.

The element:

$$\dagger = -J_1 \otimes J_1 - J_2 \otimes J_2 + J_3 \otimes J_3$$

is independent of the choice of the basis $\{J_1, J_2, J_3\}$ and acts naturally on Λ^2 with $\dagger^2 = 2\dagger + 3$. The eigenspace decomposition:

$$\Lambda^2 = \{\dagger = 3\} \oplus \{\dagger = -1\} \tag{5.29}$$

is a paraquaternionic invariant in the sense that it is preserved by $GL(n, \tilde{H})Sp(1, \mathbb{R})$ and therefore it is a hyper-paracomplex invariant preserving by $GL(n, \tilde{H})$.

5.3. *The hyper-paracomplex differential*

Studying the action of $GL(n, \tilde{H})$ on the bundle Λ^k, we consider the subbundle:

$$A^k = \sum_{I \in S_1^2(-1)} (\Lambda_I^{k,0} \oplus \Lambda_I^{0,k}).$$

It is not difficult to see that

$$A^k = \sum_{P \in S_1^2(1)} (\Lambda_P^{k,0} \oplus \Lambda_P^{0,k}).$$

Indeed, any 2-form $\omega \in \Lambda^2$ decomposes according to (5.29):

$$\begin{aligned}\omega(X,Y) = {} & \tfrac{1}{4}\{3\omega(X,Y) + \omega(J_1X, J_1Y) + \omega(J_2X, J_2Y) - \omega(J_3X, J_3Y)\} \\ & + \tfrac{1}{4}\{\omega(X,Y) - \omega(J_1X, J_1Y) - \omega(J_2X, J_2Y) + \omega(J_3X, J_3Y)\}\end{aligned} \tag{5.30}$$

For example,

$$A^2 = \Lambda^{2,0}_{J_3} \oplus \Lambda^{0,2}_{J_3} \oplus A^{1,1}_{J_3} = \Lambda^{2,0}_{J_2} \oplus \Lambda^{0,2}_{J_2} \oplus A^{1,1}_{J_2},$$

where

$$A^{1,1}_{J_3} = \{\omega \in \Lambda^2 : J_3\omega = \omega \quad \text{and} \quad J_2\omega = \omega\},$$
$$A^{1,1}_{J_2} = \{\omega \in \Lambda^2 : J_2\omega = -\omega \quad \text{and} \quad J_3\omega = -\omega\}.$$

If g is a hyper-parahermitian metric then the Kähler form F_3 (resp., F_2) is a smooth section of $A^{1,1}_{J_3}$ (resp., $A^{1,1}_{J_2}$) and conversely any smooth section F_3 of $A^{1,1}_{J_3}$ (resp., F_2 of $A^{1,1}_{J_2}$) defines an (possibly degenerate) hyper-parahermitian metric $g = -F_3(., J_3.)$ (resp., $g = F_2(., .J_2)$). We will call such a form *a hyper-paracomplex (1, 1)-form*.

There is a projection $\eta : \Lambda^k \to A^k$, whose kernel is the subbundle:

$$B^k = \bigcap_{I \in S^2_1(-1)} (\Lambda^{k-1,1}_I \oplus \Lambda^{k-2,2}_I \oplus \cdots \oplus \Lambda^{1,k-1}_I)$$
$$= \bigcap_{P \in S^2_1(1)} (\Lambda^{k-1,1}_P \oplus \Lambda^{k-2,2}_P \oplus \cdots \oplus \Lambda^{1,k-1}_P).$$

In particular, the two eigenspaces of the operator † are related with A^2, B^2 as follows:

$$A^2 = \{\dagger = -1\}, \quad B^2 = \{\dagger = 3\}.$$

The projections ω^{A^2} and ω^{B^2} are given by:

$$\omega^{A^2}(X, Y) = \tfrac{1}{4}\{3\omega(X, Y) + \omega(J_1X, J_1Y) + \omega(J_2X, J_2Y) - \omega(J_3X, J_3Y)\},$$
$$\omega^{B^2}(X, Y) = \tfrac{1}{4}\{\omega(X, Y) - \omega(J_1X, J_1Y) - \omega(J_2X, J_2Y) + \omega(J_3X, J_3Y)\}.$$

We define the *hyper-paracomplex differential*:

$$D : A^k \to A^{k+1}$$

simply by composition of the projection η end the exterior differential d:

$$D = \eta \circ d.$$

For example, if ω is a 1-form, then,

$$D\omega = (d\omega)^{2,0}_{J_3} + (d\omega)^{0,2}_{J_3} + \tfrac{1}{2}((d\omega)^{1,1}_{J_3} + J_2(d\omega)^{1,1}_{J_3})$$
$$= (d\omega)^{2,0}_{J_2} + (d\omega)^{0,2}_{J_2} + \tfrac{1}{2}((d\omega)^{1,1}_{J_2} - J_3(d\omega)^{1,1}_{J_2}) \tag{5.31}$$

Theorem 5.6. *An almost hyper-paracomplex structure is integrable if and only if* $D^2 = 0$.

Proof. The condition $D^2 = 0$ is equivalent to the assertion that the exterior derivative of a 2-form of type (1, 1) relative to all $I \in S^2_1(-1)$ and all $P \in S^2_1(1)$ has no (0, 3)+(3, 0)-component relative to any $I \in S^2_1(-1)$ and to any $P \in S^2_1(1)$. The latter condition holds on a hyper-paracomplex manifold since all almost complex structures of the two-sheeted hyperboloid $S^2_1(-1)$ as well as all paracomplex structures of the one-sheeted hyperboloid $S^2_1(1)$ are integrable due to the existence of a linear identity between their Nijenhuis tensors [21,7].

To prove the converse, let Ω be (1, 1)-form with respect to the almost complex structure J_3, $\Omega \in \Lambda^{1,1}_{J_3}$. The 2-form C defined by $C(X, Y) = \Omega(X, Y) - \Omega(J_1X, J_1Y)$ belongs to B^2. The condition $DC = 0$ is equivalent to the relation:

$$dC(X, Y, Z) = dC(IX, IY, Z) + dC(IX, Y, IZ) + dC(X, IY, IZ)$$

for all $I \in S^2_1(-1)$ and

$$-dC(X, Y, Z) = dC(PX, PY, Z) + dC(PX, Y, PZ) + dC(X, PY, PZ)$$

for all $P \in S^2_1(1)$. In particular,

$$-dC(J_3X, J_3Y, J_3Z) = dC(J_1X, J_1Y, J_3Z) + dC(J_1X, J_3Y, J_1Z) + dC(J_3X, J_1Y, J_1Z). \quad (5.32)$$

Let ∇^{CP} be a complex product connection, $\nabla^{CP} J_a = 0$, with torsion T^{CP}. Then, the Nijenhuis tensors are related with T^{CP} as follows:

$$N_a = -T^{CP}(J_a., J_a.) - J_a^2 T^{CP}(., .) + J_a T^{CP}(J_a., .) + J_a T^{CP}(., J_a.). \quad (5.33)$$

Use (5.33) to express the exterior derivative of a 2-form as:

$$dC(X, Y, Z) = \nabla^{CP} C(X; Y, Z) + \nabla^{CP} C(Y; Z, X) + \nabla^{CP} C(Z; X, Y) + C(T^{CP}(X, Y), Z) + C(T^{CP}(Y, Z), X) + C(T^{CP}(Z, X), Y). \quad (5.34)$$

Insert (5.34) into (5.32) and use (5.33) to get:

$$\Omega(J_3N_2(J_3X, J_3Y), Z) + \Omega(J_3N_2(J_3Y, J_3Z), X) + \Omega(J_3N_2(J_3Z, J_3X), Y) + \Omega(J_1N_2(J_3X, J_3Y), J_2Z) + \Omega(J_1N_2(J_3Y, J_3Z), J_2X) + \Omega(J_1N_2(J_3Z, J_3X), J_2Y) = 0 \quad (5.35)$$

valid for any (1, 1)-form with respect to J_3. In particular, take $\Omega = Z \wedge J_3Z$, we get from (5.35) that $N_2 = 0$.

Similarly, we obtain $N_1 = 0$. Hence, the almost hyper-paracompex structure is integrable [21]. □

Further, we need two lemmas:

Lemma 5.7. *Let $F \in A^{1,1}_{J_3}$ be a non-degenerate hyperparacomplex (1, 1)-form on a hyperparacomplex manifold (M, J_1, J_2, J_3). The metric $g = -F(., J_3.)$ is HPKT-metric if and only if F is D-closed, $DF = 0$.*

Such a form is called a HPKT-form.

Proof. Suppose that g is HPKT. For any complex structure $I \in S^2_1(-1)$ (paracomplex strucrure $P \in S^2_1(1)$) the form dF_I (dF_P) has type $(2, 1) + (1, 2)$ with respect to the complex structure I (paracomplex structure P). But since $d_1F_1 = d_2F_2 = d_3F_3$ we deduce that dF_I (dF_P) has type $(2, 1) + (1, 2)$ with respect to the two paracomplex structures and the complex structure: $dF_I \in B^3$ ($dF_P \in B^3$) that is $DF_I = 0$ ($DF_P = 0$). Since $F_I = F$ ($F_P = F$) we obtain the result.

Suppose now that $DF = 0$. The condition $F \in A^{(1,1)}_{J_3}$ also reads as:

$$F(X, Y) = F(J_3X, J_3Y) = F(J_1X, J_1Y),\ F(J_3X, J_1Y) = F(J_1X, J_3Y). \tag{5.36}$$

Use the torsion free complex product connection, (5.34), (5.36) and (5.32), to get:

$$\begin{aligned}-dF(J_3X, J_3Y, J_3Z) &= \nabla^{CP}F(J_3X; Y, Z) + \nabla^{CP}F(J_3Y; Z, X) + \nabla^{CP}F(J_3Z; X, Y)\\ &\quad + 2\nabla^{CP}F(J_1X; J_1Y, J_3Z) + 2\nabla^{CP}F(J_1Y; J_1Z, J_3X)\\ &\quad + 2\nabla^{CP}F(J_1Z; J_1X, J_3Y).\end{aligned}$$

Consequently,

$$\begin{aligned}-dF(J_3X, J_3Y, J_3Z) &= \nabla^{CP}F(J_1X; J_1Y, J_3Z) + \nabla^{CP}F(J_1Y; J_1Z, J_3X)\\ &\quad + \nabla^{CP}F(J_1Z; J_1X, J_3Y).\end{aligned}$$

Define $G = F(\cdot, J_2\cdot)$, we have the following sequence of equalities

$$\begin{aligned}-dG(J_1X, J_1Y, J_1Z) &= -\nabla^{CP}G(J_1X; J_1Y, J_1Z) - \nabla^{CP}G(J_1Y; J_1Z, J_1X)\\ &\quad - \nabla^{CP}G(J_1Z; J_1X, J_1Y)\\ &= \nabla^{CP}F(J_1X; J_1Y, J_3Z) + \nabla^{CP}F(J_1Y; J_1Z, J_3X)\\ &\quad + \nabla^{CP}F(J_1Z; J_1X, J_3Y)\\ &= -dF(J_3X, J_3Y, J_3Z)\end{aligned}$$

and therefore $dF(J_3X, J_3Y, J_3Z) = dG(J_1X, J_1Y, J_1Z)$. In other words, $d_3F_3 = d_1F_1$ with $F_3 = F$ and $F_1 = G$. □

Lemma 5.8. *A PHKT-metric locally admits a potential if and only if the corresponding HPKT-form is locally D-exact.*

Proof. Suppose that $F = -\frac{1}{2}(dd_3 + d_1d_2)\mu$. Then, $F = \frac{1}{2}(d\theta + J_1d\theta)$ with $\theta = -J_3d\mu = -d_3\mu$. Note that $d\theta$ is (1, 1)-form (for J_3) since $d\theta = -dd_3\mu$. Therefore, $F = D\theta$ according to (5.31).

Conversely, suppose that $F = D\theta$ for some 1-form θ. Since F is a $(1,1)$-form for J_3, we obtain from (5.31)

$$d\theta \in \Lambda^{(1,1)}_{J_3}, \quad F = \tfrac{1}{2}(d\theta + J_1 d\theta).$$

Since J_3 is an integrable complex structure, the local dd_3-lemma holds: locally there exists μ such that $d\theta = -dd_3\mu$. We get then

$$F = \tfrac{1}{2}(d\theta + J_1 d\theta) = -\tfrac{1}{2}(dd_3 + d_1 d_2)\mu. \quad \square$$

Theorem 5.9. *On an* $(4n \geq 8)$*-dimensional manifold any HPKT-metric admits locally an HPKT-potential or equivalently any D-closed HPKT-form is locally D-exact.*

Proof. Any $(4n \geq 8)$-dimensional HPKT-manifold is a manifold with a structure group contained in $Sp(n,\mathbb{R}) \subset GL(n,\tilde{H})$ and it is 1-integrable due to the existence of a torsion-free $GL(n,\tilde{H})$-connection, the complex product connection [2,3]. Therefore, since the operator D is $GL(n,\tilde{H})$-invariant, it is sufficient to show that any D-closed HPKT-form is locally D-exact in $\mathbb{R}^{4n}$ with the standard HPKT-structure.

In $\mathbb{R}^{4n}$ we split the complex coordinates into two sets $(\{z^j, w^j\}, j = 1,\ldots,n)$. The hyper-paracomplex structure is given by:

$$J_1 = -dz^j \otimes \frac{\partial}{\partial z^j} - dw^j \otimes \frac{\partial}{\partial w^j} + d\overline{z}^j \otimes \frac{\partial}{\partial \overline{z}^j} + d\overline{w}^j \otimes \frac{\partial}{\partial \overline{w}^j},$$
$$J_2 = id\overline{w}^j \otimes \frac{\partial}{\partial z^j} - id\overline{z}^j \otimes \frac{\partial}{\partial w^j} + idw^j \otimes \frac{\partial}{\partial \overline{z}^j} - idz^j \otimes \frac{\partial}{\partial \overline{w}^j},$$
$$J_3 = id\overline{w}^j \otimes \frac{\partial}{\partial z^j} - id\overline{z}^j \otimes \frac{\partial}{\partial w^j} - idw^j \otimes \frac{\partial}{\partial \overline{z}^j} + idz^j \otimes \frac{\partial}{\partial \overline{w}^j}.$$

Let $F_3 \in \Lambda^{1,1}_{J_3}$, $DF_3 = 0$. The hyper-parahermitian condition for the 2-tensor $h = F_3(.,J_3)$ implies that

$$h_{z^j\overline{z}^k} = h_{w^k\overline{w}^j}; \quad h_{z^j\overline{w}^k} = -h_{z^k\overline{w}^j}. \tag{5.37}$$

The condition $d_1F_1 = d_3F_3$ becomes:

$$\begin{aligned} &h_{z^j\overline{w}^k,\overline{w}^l} + h_{z^k\overline{w}^l,\overline{w}^j} + h_{z^l\overline{w}^j,\overline{w}^k} = 0, \quad h_{w^j\overline{z}^k,w^l} + h_{w^k\overline{z}^l,w^j} + h_{w^l\overline{z}^j,w^k} = 0\\ &h_{w^j\overline{z}^k,\overline{z}^l} + h_{w^k\overline{z}^l,\overline{z}^j} + h_{w^l\overline{z}^j,\overline{z}^k} = 0, \quad h_{z^j\overline{w}^k,z^l} + h_{z^k\overline{w}^l,z^j} + h_{z^l\overline{w}^j,z^k} = 0,\\ &h_{z^j\overline{z}^l,\overline{w}^k} - h_{z^k\overline{z}^l,\overline{w}^j} - h_{z^j\overline{w}^k,\overline{z}^l} = 0, \quad h_{z^j\overline{z}^k,w^l} - h_{z^j\overline{z}^l,w^k} + h_{w^k\overline{z}^l,z^j} = 0,\\ &h_{z^j\overline{z}^k,\overline{z}^l} - h_{z^j\overline{z}^l,\overline{z}^k} - h_{w^k\overline{z}^l,\overline{w}^j} = 0, \quad h_{z^j\overline{z}^l,z^k} - h_{z^k\overline{z}^l,z^j} + h_{z^j\overline{w}^k,w^l} = 0. \end{aligned} \tag{5.38}$$

The first and second lines of (5.38), when combined with the antisymmetry in j,k of $h_{z^j\overline{w}^k}$, allow us to apply the local $\partial\overline{\partial}$-lemma. Therefore, we can write:

$$h_{z^j\overline{w}^k} = (\partial_{z^j}\partial_{\overline{w}^k} - \partial_{z^k}\partial_{\overline{w}^j})\mu; \quad h_{w^j\overline{z}^k} = (\partial_{w^j}\partial_{\overline{z}^k} - \partial_{w^k}\partial_{\overline{z}^j})\mu, \tag{5.39}$$

where μ is some (real) (by para-hermiticity of the metric-and therefore identical in the two equations (5.39)) function. Inserting (5.39) into the third equation of (5.38) gives:

$$\partial_{\overline{w}^k}(h_{z^j\overline{z}^l} - \mu,_{z^j\overline{z}^l}) - \partial_{\overline{w}^j}(h_{z^k\overline{z}^l} - \mu,_{z^k\overline{z}^l}) = 0, \tag{5.40}$$

and therefore,

$$h_{z^j\bar{z}^k} = \mu,_{z^j\bar{z}^k} + \partial_{\bar{w}^j}\alpha_{\bar{z}^k} \tag{5.41}$$

for some integration one form $\alpha_{\bar{z}^k}$. Combining this with the fourth equation of (5.38) gives $\alpha_{\bar{z}^k} = \mu,_{w^k}$. Thus, we get that the function μ generates F_3. The Lemma 5.8 completes the proof. □

Remark 5.10. A hyperbolic version of Salamon's quaternionic operator [30]. We recall that an *almost paraquaternionic structure* on M is a rank-3 subbundle $P \subset End(TM)$ which is locally spanned by an almost hyper-paracomplex structure $\tilde{H} = (J_a)$. Equivalently, the structure group of TM can be reduced to $GL(n, \tilde{H})Sp(1, \mathbb{R})$. A linear connection on TM is called *paraquaternionic connection* if it preserves P. An almost paraquaternionic structure is said to be a *paraquaternionic* if there is a torsion-free paraquaternionic connection. The paraquaternionic condition controls the Nijenhuis tensors in the sense that $N(X, Y)(J_a) := N_a$(X,Y) preserves the subbundle P. When $n \geq 2$, the paraquaternionic condition is a strong condition which is equivalent to the 1-integrability of the associated $GL(n, \tilde{H})Sp(1, \mathbb{R})$-structure [2,3]. We can extend the hyper-paracomplex operator D defining it on an almost paraquaternionic manifold locally in the same way. Consequently, Theorem 5.6 is also true, namely an almost paraquaternionic manifold is paraquaternionic exactly when $D^2 = 0$. The proof of Theorem 5.6 goes through in this case also. Now the $Sp(1, \mathbb{R})$-part of the paraquaternionic connection used, adds an additional $Sp(1, \mathbb{R})$ term in formula (5.33) which reflects on (5.35), whence the Nijenhuis tensors preserve the subbundle P. Using the 1-integrability of the paraquaternionic structure and the proof of Theorem 5.9 one gets the local exactness of certain D-closed forms.

Acknowledgements

The research is partially supported by Contract MM 809/1998 with the Ministry of Science and Education of Bulgaria, Contracts 586/2002 and 35/2003 with the University of Sofia "St. Kl. Ohridski". S.I. is a member of the EDGE, Research Training Network HPRN-CT-2000-00101, supported by the European Human Potential Programme. We would like to thank to T. Gramtchev for his valuable comments.

References

[1] M. Akivis, S. Goldberg, Conformal Differential Geometry and its Generalizations, Wiley, 1996.
[2] A. Andrada, Complex product structures on differentiable manifolds, preprint.
[3] A. Andrada, S. Salamon, Complex product structures on Lie algebras, Forum. Math. 17 (2005) 261–295.
[4] B. Banos, A. Swann, Potentials for hyper-Kähler metrics with torsion, Classical Quantum Gravity 21 (2004) 3127–3135.
[5] L. Barberis, I.D. Miatelo, R.J. Miatelo, On certain locally homogeneous Clifford manifolds, Ann. Glob. Anal. Geom. 13 (1995) 289–301.
[6] J. Barret, G.W. Gibbons, M.J. Perry, C.N. Pope, P. Ruback, Kleinian geometry and the $N = 2$ superstring, Int. J. Mod. Phys. A9 (1994) 1457–1494.

[7] N. Blazic, S. Vukmirovic, Four-dimensional Lie algebras with para-hypercomplex structure, math.DG/0310180.
[8] A. Borel, Clifford-Klein forms of symmetric spaces, Topology 2 (1963) 111–122.
[9] M. Dunajski, Hyper-complex four-manifolds from Tzitzeica equation, J. Math. Phys. 43 (2002) 651–658.
[10] A. Fino, H. Pedersen, Y.-S. Poon, M.W. Sorensen, Neutral Calabi–Yau structures on Kodaira manifolds, Commun. Math. Phys. 248 (2004) 255–288.
[11] Th. Friedrich, S. Ivanov, Parallel spinors and connections with skew-symmetric torsion in string theory, Asian J. Math. 6 (2002) 303–336.
[12] E. Garcia-Rio, Y. Matsushita, R. Vasquez-Lorenzo, Paraquaternionic Kähler manifold, Rocky Mountain J. Math. 31 (2001) 237–260.
[13] P. Gauduchon, Hermitian connections and Dirac operators, Boll. Un. Mat. Ital. B (7) 11 (Suppl. 2) (1997) 257–288.
[14] G. Grantcharov, Y.S. Poon, Geometry of hyper-Kähler connections with torsion, Comm. Math. Phys. 213 (2000) 19–37.
[15] N. Hitchin, Hypersymplectic quotients, Acta Acad. Sci. Tauriensis 124 (Suppl.) (1990) 169–180.
[16] P.S. Howe, G. Papadopoulos, Twistor spaces for hyper-Kähler manifolds with torsion, Phys. Lett. B379 (1996) 80–86.
[17] L. Hörmander, The Analysis of Linear Partial Differential Operators, I–IV, Springer Verlag, Berlin, 1983–85.
[18] C.M. Hull, Actions for (2,1)sigma models and strings, Nucl. Phys. B509 (1) (1988) 252–272.
[19] C.M. Hull, A geometry for non-geometric string backgrounds, hep-th/0406102.
[20] S. Ivanov, V. Tsanov, Complex product structures on some simple Lie groups, Diff. Geom. Appl., in press.
[21] S. Ivanov, S. Zamkovoy, Para-hermitian and para-quaternionic manifolds, math.DG/0310415.
[22] H. Kamada, Neutral hyper-Kähler structures on primary Kodaira surfaces, Tsukuba J. Math. 23 (1999) 321–332.
[23] P. Libermann, Sur le probleme d'equivalence de certains structures infinitesimales, Ann. Math. Pure Appl. 36 (1954) 27–120.
[24] M. Mamone Capria, S. Salamon, Yang-Mills fields on quaternionic spaces, Nonlinearity 1 (1988) 517–530.
[25] J. Michelson, A. Strominger, The geometry of (super) conformal quantum mechanics, Comm. Math. Phys. 213 (2000) 1–17.
[26] J. Milnor, Curvatures of left invariant metrics on Lie groups, Adv. Math. 21 (3) (1976) 293–329.
[27] H. Ooguri, C. Vafa, Geometry of $N = 2$ strings, Nucl. Phys. B361 (1991) 469–518.
[28] P.K. Rashevskii, The scalar field in a stratified space, Trudy Sem. Vekt. Tenz. Anal. 8 (1948) 225–248 (in Russian).
[29] F. Raymond, A. Vasquez, 3-manifolds whose universal coverings are Lie groups, Topology Appl. 12 (1981) 161–179.
[30] S. Salamon, Differential geometry of quaternionic manifolds, Ann. Sci. École Norm. Sup. (4) 19 (1986) 31–55.
[31] S. Salamon, Complex structures on nilpotent Lie algebras, J. Pure Appl. Algebra 157 (2001) 311–333.
[32] P. Scott, The geometries of 3-manifolds, Bull. Lond. Math. Soc. 15 (1983) 407–487.
[33] M. Verbitski, HyperKähler manifolds with torsion, supersymmetry and Hodge theory, Asian J. Math. 6 (2002) 679–712.
[34] S. Vukmirovic, Paraquaternionic reduction, math.DG/0304424.